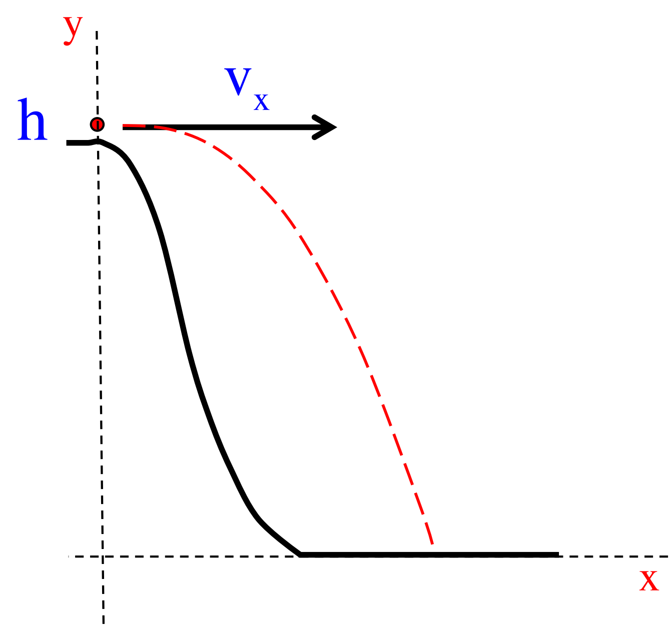
Some Slightly Exotic Calculus

Let's say you've thrown a ball off a cliff that is *h* meters high. Ignoring air friction, the ball's velocity in the x-direction is observed to be v_x (this will not change as there are no frictional effects to make the change). The red dashed line in the time lapse photo to the right shows the motion.



The relationship between the *y-coordinate* and the *x-coordinate* just happens to be:

$$y = h - kx^2$$

where *h* is the height of the cliff and *k* is a constant related to gravity. (In fact, this is the equation for a downward pointing parabola.)

In terms of time, the *x-coordinate* of the body is: $x = v_x t$

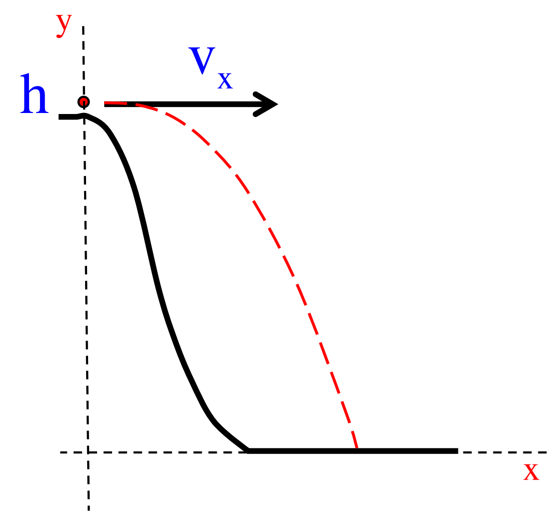
where, again, v_x is the constant velocity of the body in the x-direction.

Insert a.)

So here's the question: With

$$x = v_x t \quad \text{and} \quad y = h - kx^2$$

what is the ball's velocity in the *y-direction* in terms of *x*?



Knowing that the velocity in the *y-direction* is the time derivative of the *y-coordinate function*, or

$$v_y = \frac{dy}{dt}$$

the chintzy way to do this would be to substitute the time-related *x-coordinate* function into the *x-related* y-coordinate function, take the time derivative, then substitute *x information* back into that result.

That is: If $y = h - kx^2$ and $x = v_x t$, then by substitution $y = h - k(v_x t)^2$, we can write:

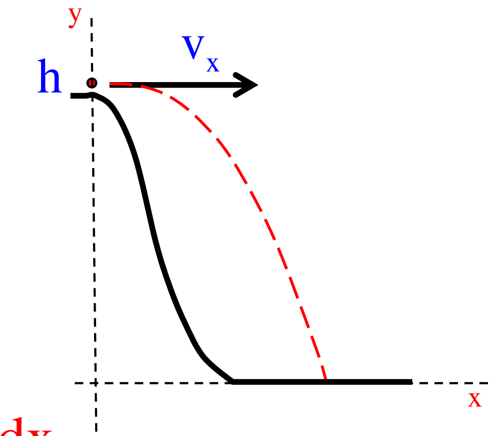
$$v_y = \frac{dy}{dt} = \frac{d\left(h - k(v_x t)^2\right)}{dt}$$

$$= -kv_x^{\ 2}(2t) = -2kv_x(v_x t)$$

$$= -2kv_x x$$

Insert b.)

The more elegant way to do this is to utilize what is called *the chain rule*. It states (loosely) that if you want the derivative of a function that you know in terms of another function, you can get that result by executing the operation:



$$v_y = \frac{df(y(x(t)))}{dt} = \frac{df(y(x))}{dx}\frac{df(x(t))}{dt} \quad \text{or just} \quad v_y = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

(Notice how the "dx's" might be seen to cancel, if they could, leaving us with $v_y = {dy}/{dt}$ . . . cool, eh?)

Executing that operation, remembering that,

$x = v_x t$ and $y = h - kx^2$

and we get:

$$\begin{aligned} v_y &= \frac{dy}{dt} \\ &= \frac{dy}{dx}\frac{dx}{dt} \\ &= \left(\frac{d(h-kx^2)}{dx}\right)\left(\frac{d(v_x t)}{dt}\right) \\ &= -2kxv_x \end{aligned}$$

. . . much more satisfying.

insert c)